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Typical Cluster Size for Two-Dimensional Percolation Processes

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The typical cluster size for two-dimensional percolation models is discussed. It is shown that, for $W_0 = \{x \in Z^2: 0 \to x\}$, $[-\lim_{n\to\infty}(1/n)\log P_p(|W_0| = n)]^{-1} \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$, provided that $E_p(|W_0|^2)/E_p(|W_0|) \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$. Furthermore, we introduce a new quantity $f_s(p)$, which may be thought of as the singular part of the free energy, and show that $f_s(p) \approx |p - p_c|^{2\nu}$ provided that the correlation length $\approx |p - p_c|^{-\nu}$ as $p \to p_c$.

KEY WORDS: Percolation; typical cluster size; singular part of the free energy.

1. INTRODUCTION

The pursue of this paper is to discuss some characteristics of the typical cluster size for the two-dimensional percolation models satisfying the fundamental assumption as in Ref. 4, p. 371. For simplicity we only describe our results for the site percolation model on Z^2 and leave the task of extending our discussion to general models to the reader. Let us now introduce the two-dimensional site percolation model. Let P_p denote the probability measure under which all sites of the lattice Z^2 are independently occupied (nonoccupied) with probability p (respectively $1 - p$). We say that x is connected to y if there is a nearest neighbor path over occupied sites connecting x and y. We denote this event as $\{x \to y\}$. Let $W_0 = \{x \in Z^2: 0 \to x\}$: the cluster of occupied sites connected to 0. This paper is devoted to the study of certain special properties of the "typical cluster size" about the critical point $p_c = \inf\{p: P_p(0 \to \infty) > 0\}$. In the

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paper, "Scaling Theory of Percolation Clusters" [1979], Stauffer[8] introduced the following basic postulate: "We assume that the critical behavior of percolation is dominated by clusters of size $S_\xi \propto |p - p_c|^{-1/\sigma}$, where differently defined typical cluster sizes S_ξ all diverge with the same exponent."

Furthermore, he also proposed a scaling hypothesis, which seems to suggest that

$$P_p(|W_0| = n) \propto \begin{cases} n^{1-\tau} \exp[-n/S_\xi(p)] & \text{for} \quad p < p_c \qquad (*) \\ n^{1-\tau} \exp\{-[n/S_\xi(p)]^{1/2}\} & \text{for} \quad p > p_c \qquad (**) \end{cases}$$

This suggests that "typical clusters" are clusters of size $\sim |p - p_c|^{-1/\sigma}$ and the critical exponent $1/\sigma$ is equal to the exponent Δ of the gaps

$$S_t(p) \equiv E_p(|W_0|^{t+1}; |W_0| < \infty)/E_p(|W_0|^t; |W_0| < \infty)$$

To see why $1/\sigma = \Delta$, we apply the scaling hypothesis to obtain, for $p < p_c$,

$$\begin{aligned} S_t(p) &= \frac{\sum_{n=0}^{\infty} n^{t+1} P_p(|W_0| = n)}{\sum_{n=0}^{\infty} n^t P_p(|W_0| = n)} \\ &\propto \frac{\sum_{n=0}^{\infty} n^{t+1+(1-\tau)} \exp[-n/S_\xi(p)]}{\sum_{n=0}^{\infty} n^{t+(1-\tau)} \exp[-n/S_\xi(p)]} \\ &\propto \frac{S_\xi(p)^{t+2+(1-\tau)} \int_0^\infty x^{t+1+(1-\tau)} e^{-x}\, dx}{S_\xi(p)^{t+1+(1-\tau)} \int_0^\infty x^{t+(1-\tau)} e^{-x}\, dx} \\ &\propto S_\xi(p) \end{aligned}$$

and similarly for $p > p_c$,

$$\begin{aligned} S_t(p) &\propto \frac{\sum_{n=0}^{\infty} n^{t+1+(1-\tau)} \exp\{-[n/S_\xi(p)]^{1/2}\}}{\sum_{n=0}^{\infty} n^{t+(1-\tau)} \exp\{-[n/S_\xi(p)]^{1/2}\}} \\ &\propto \frac{S_\xi(p)^{t+2+(1-\tau)} \int_0^\infty x^{t+1+(1-\tau)} \exp(-\sqrt{x})\, dx}{S_\xi(p)^{t+1+(1-\tau)} \int_0^\infty x^{t+(1-\tau)} \exp(-\sqrt{x})\, dx} \\ &\propto S_\xi(p) \end{aligned}$$

On the other hand, from the scaling hypothesis, we can also see that

$$-\frac{1}{n} \log P_p(|W_0| = n) \propto S_\xi(p)^{-1} - (1-\tau)\, n^{-1} \log n \qquad \text{for} \quad p < p_c$$

$$-\frac{1}{\sqrt{n}} \log P_p(|W_0| = n) \propto S_\xi(p)^{-1/2} - (1-\tau)\, n^{-1/2} \log n \qquad \text{for} \quad p > p_c$$

Letting $n \to \infty$, we have

$$S_{\rm I}(p)^{-1} \equiv -\lim_{n\to\infty} \frac{1}{n} \log P_p(|W_0| = n) \propto S_\xi(p)^{-1} \qquad \text{for} \quad p < p_c$$

$$S_{\rm II}(p)^{-1/2} \equiv -\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log P_p(|W_0| = n) \propto S_\xi(p)^{-1/2} \qquad \text{for} \quad p > p_c$$

Guided by scaling theory, we want to find a quantity $S_\xi(p)$ associated with the correlation length $\xi(p)$ such that $S_\xi(p)$ diverges at the same rate as $S_t(p)$ and $S_{\rm I}(p)$ for $p < p_c$, or $S_t(p)$ and $S_{\rm II}(p)$ for $p > p_c$. We call such a quantity $S_\xi(p)$ the typical cluster size associated with the correlation length $\xi(p)$.

The concept of correlation length has been well studied. The usual definitions for the correlation length are

$$\xi(p) = \inf\{N\text{: for every } x \in Z^2,\ P_p(0 \to x) \leqslant \exp(-|x|/N)\} \qquad \text{for} \quad p < p_c$$

$$\xi_t(p) = \left[\sum_{x \in Z^2} |x|^t P_p(0 \to x; 0 \nrightarrow \infty) \Big/ \sum_{x \in Z^2} P_p(0 \to x; 0 \nrightarrow \infty) \right]^{1/t} \qquad \text{for} \quad p < p_c \quad \text{or} \quad p > p_c$$

$$L(p, \varepsilon) = \begin{cases} \min\{n\text{: } CR_p(n) \leqslant \varepsilon\} & \text{for} \quad p < p_c \\ \min\{n\text{: } CR_p(n) \geqslant 1 - \varepsilon\} & \text{for} \quad p > p_c \end{cases}$$

where $CR_p(n) = P_p$ (there exists an occupied crossing from left to right of $B(n) \equiv \{x \in Z^2\text{: } |x| \leqslant n\}$: the box of size n centered at 0). In the definition of $L(p, \varepsilon)$ it is not important to choose a precise value of ε, since we can show that for ε smaller than some small enough ε_0, all the above definitions of the correlation length are equivalent in the sense that if $\xi(p) \approx |p - p_c|^{-\nu}$ as $p \downarrow p_c$ or $\xi(p) \approx |p_c - p|^{-\nu}$ as $p \uparrow p_c$, then so do the others. Here we denote $f(p) \approx |p_c - p|^\lambda$ as $p \uparrow p_c$ if

$$\lim_{p \uparrow p_c} \frac{\log f(p)}{\log |p_c - p|} = \lambda$$

and similarly, $f(p) \approx |p - p_c|^\lambda$ as $p \downarrow p_c$ if

$$\lim_{p \downarrow p_c} \frac{\log f(p)}{\log |p - p_c|} = \lambda$$

From now on we shall fix ε and write $L(p)$ instead of $L(p, \varepsilon)$. For further details on the correlation length we refer to Refs. 1, 5, and 7. Since these definitions of the correlation length are equivalent, it is not necessary to distinguish them. For percolation in two dimensions it would be more convenient to work with the $L(p)$. Let us consider

$$S_L(p) \equiv E_p[\# \{x \in B(L): x \to \partial B(L)\}]$$

This quantity has already been studied extensively by Kesten[5] and was shown to play a very important role in the proofs of scaling relations for the critical exponents which were predicted by the scaling hypothesis mentioned above. Kesten's[5] work provides us with a tool to study the typical cluster size rigorously without the need of using the scaling assumptions (*), (**). In fact, we shall define $S_L(p)$ as the typical cluster size associated with the correlation length $L(p)$. To some extent we may think of a "typical cluster" as the union of the clusters that connect to the boundary $\partial B(L)$ of the box $B(L)$ by occupied paths. To justify the definition of $S_L(p)$ as the typical cluster size, we need to show that $S_L(p)$ is of the same order as $S_t(p)$, $S_{\mathrm{I}}(p)$, or $S_{\mathrm{II}}(p)$. Half of this was shown in (1.1) of the following result:

Theorem (Kesten). Let $L(p)$ be the correlation length defined as before and let $\pi_p(L) = P_p$(0 is connected to a vertical line at distance $L(p)$ away from the origin). Then

$$S_t(p) \asymp S_L(p) \asymp \pi_p(L)\, L^2(p) \qquad \text{for} \quad t > 1/3 \tag{1.1}$$

$$E_p(|W_0|; |W_0| < \infty) \asymp \pi_p^2(L)\, L^2(p) \tag{1.2}$$

There exists a positive constant δ such that

$$P_p(\infty > |W_0| \geqslant \delta S_L(p)) \geqslant \tfrac{1}{2}\pi_p(L) \tag{1.3}$$

Here $g(p) \asymp h(p)$ means that there are positive constants A, $\tilde{A}$ independent of p such that $Ag(p) \leqslant h(p) \leqslant \tilde{A}g(p)$. For the proof of this theorem we refer the reader to Refs. 4 and 5.

In this paper we add strength to the concept of the typical cluster size $S_L(p)$ by showing (see Section 3) the following:

Proposition 1. Assume that $S_L(p) \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$. Then also, $S_{\mathrm{I}}(p) \approx |p - p_c|^{-\Delta}$ as $p \uparrow p_c$.

Note that the limit in the definition of $S_{\mathrm{I}}(p)$ exists by the supermultiplicative property of $n^{-1}P_p(|W_0| = n)$ (see Ref. 6).

The proof of the above result will be based on the following:

Lemma. Let $M_t[L(p)]$ = the tth moment of the number of sites in the box $B(L)$ connected to its boundary $\partial B(L)$ by occupied paths, i.e.,

$$M_t[L(p)] = E_p[|\{x \in B(L): x \to \partial B(L)\}|^t]$$

Then

$$M_t[L(p)] \leqslant B_t[K_1 S_L(p)]^t$$

where $B_t = (t+1)!$ and K_1 is a positive constant depending only on ε.

The proof of the above lemma can be found in Ref. 4 except the fact that $B_t = (t+1)!$. In our opinion it is not easy to see that the B_t are of order $(t+1)!$ therein, since its proof is based on a rather complicated combinatorial argument. Since our proof for Proposition 1 depends on the B_t, we shall give a proof for the lemma in Section 2 with an inductive argument.

Remarks:

1. If we apply the estimate in the lemma to the argument in Section 3 of Ref. 5 we can show that for $p > p_c$

$$E_p(|W_0|^t;\, |W_0| < \infty) \leqslant C_t[K_2 S_L(p)]^t\, \pi_p(L)$$

where $C_t = (3t)!$. However, the constants $C_t = (3t)!$ are not of the right order, since it was conjectured by Stauffer[8] that $C_t = t!$ for $p < p_c$ and that $C_t = (2t)!$ for $p > p_c$.

2. At this point we do not know how to show that $S_L(p)$ is of the same order as $S_{\mathrm{II}}(p)$ for $p > p_c$ and we cannot even show the existence of the limit as in the definition of $S_{\mathrm{II}}(p)$.

Having discussed several ways to look at the typical cluster size, we next want to study its role in the singular behavior of the free energy,

$$f(p) = \sum_{n \geqslant 1} \frac{1}{n} P_p(|W_0| = n)$$

It is well known that the free energy can be shown to be the same as the number of clusters per site. It was conjectured in Ref. 9 that the free energy is singular at p_c. It is not clear at all that the free energy has any singularity, since Kesten[3] showed that it is twice differentiable. The numerical calculations together with scaling theory suggested that the third derivative of the free energy should blow up at p_c at the rate $|p - p_c|^{-1-\alpha}$, where the critical exponent α is related to the exponent ν of the correlation length by the scaling relation (R) $2 - \alpha = d\nu$, $d = 2$: dimension. Thus, we expect that the singular part $f_{\mathrm{sing}}(p)$ of the free energy should behave as $|p - p_c|^{d\nu}$ in a neighborhood of p_c. However, it would be difficult to know the singular part, since we do not know whether the free energy has any singularity. While it is not easy to define the singular part $f_{\mathrm{sing}}(p)$, to prove the scaling relation (R) we propose a new way to look at this. It is based on the observation that if the free energy behaves singularly at p_c, then only the tail of the summation in $f(p)=\sum_{n\geqslant 1} n^{-1}P_p(|W_0|=n)$ should play an important role in this singularity. In other words, the mean number of clusters per site should be singular (if it were so!) due to the number of "large clusters." But how large should the cluster be in order for us to see the scaling relationship such as (R)? Physicists (e.g., Refs. 2 and 8) suggest that any cluster larger than the typical cluster size should be thought of as the large cluster. From this we believe that

$$f_s(p)\equiv \sum_{n\geqslant \delta S_L(p)} n^{-1}P_p(|W_0|=n)$$

where δ is some positive constant, should be thought of as a representative for the singular part of the free energy. In order to support our belief, in Section 4 we apply Kesten's theorem to give a proof of the following result:

Proposition 2. There exists a constant $\delta>0$ independent of p such that

$$f_s(p)\approx L^{-2}(p)$$

in some neighborhoods (p_1, p_c) and (p_c, p_2) of p_c.

This immediately implies the following:

Corollary. Assume that $L(p)\approx|p-p_c|^{-\nu}$ as $p\uparrow p_c$ (or $p\downarrow p_c$). Then

$$f_s(p)\approx|p-p_c|^{d\nu} \qquad \text{as} \quad p\uparrow p_c \quad (\text{or} \quad p\downarrow p_c)$$

where $d=2$: the dimension of the percolation model.

2. PROOF OF THE LEMMA

Fix ε as in the definition of $L(p,\varepsilon)$. From now on, C_ε, $\tilde{C}_\varepsilon$ will be constants depending only on ε and their values may vary from line to line. Denote $\pi_n=\pi_p(n)\equiv P_p$(0 is connected to a vertical line at distance n away from the origin). It was shown in Ref. 4 that for all $n\leqslant L(p)$,

$$\pi_n \asymp P_p(0\to\partial B(n)) \tag{2.1}$$

$$\pi_n \asymp \pi_{2n} \tag{2.2}$$

Recall that

$$M_t[L(p)]=E_p\{|\{x\in B(L)\colon x\to\partial B(L)\}|^t\}$$

We claim

$$M_{t+1}[L(p)]\leqslant C_\varepsilon(t+1)\,L(p)\left[\sum_{k=0}^{2L(p)}\pi_k\right]M_t[L(p)] \tag{2.3}$$

To prove this, we write

$$M_{t+1}[L(p)] = \sum_{x_1,\ldots,x_{t+1} \in B(L)} P_p\left(\bigcap_{i=1}^{t+1} \{x_i \to \partial B(L)\}\right)$$
$$= \sum_{k=0}^{2L(p)} \sum_{x_{t+1} \in R_k \cap B(L)}$$
$$\times \sum_{x_1,\ldots,x_t \in B(L)} P_p\left(\bigcap_{i=1}^{t} \{x_i \to \partial B(L)\}, x_{t+1} \to \partial B(L)\right)$$

where R_k is the set of all points at distance k from $\{x_1,\ldots, x_t\} \cup \partial B(L)$. For a fixed $k \geqslant 4$, we have[4]

$$P_p\left(\bigcap_{i=1}^{t} \{x_i \to \partial B(L)\}, x_{t+1} \to \partial B(L), \text{Circuit}_{x_{t+1}}(k)\right)$$
$$\leqslant P_p\left(\bigcap_{i=1}^{t} \{x_i \to \partial B(L) \text{ in } B(L) \setminus B_{x_{t+1}}(k/2)\}\right.$$
$$\left.\text{and } \{x_{t+1} \to \partial B_{x_{t+1}}(k/2)\}\right)$$

where $\text{Circuit}_{x_{t+1}}(k)$ is the event that there exists an occupied circuit in the annulus $B_{x_{t+1}}(k) \setminus B_{x_{t+1}}(k/2)$ centered at x_{t+1}. Then, by FKG, the

$$\text{LHS} \geqslant P_p\left(\bigcap_{i=1}^{t+1} \{x_i \to \partial B(L)\}\right) P_p(\text{Circuit}_{x_{t+1}}(k))$$
$$\geqslant C_\varepsilon P_p\left(\bigcap_{i=1}^{t+1} \{x_i \to \partial B(L)\}\right)$$
$$\text{RHS} = P_p\left(\bigcap_{i=1}^{t} \{x_i \to \partial B(L) \text{ in } B(L) \setminus B_{x_{t+1}}(k/2)\}\right)$$
$$\times P_p\{x_{t+1} \to \partial B_{x_{t+1}}(k/2)\}$$
$$\leqslant \tilde{C}_\varepsilon P_p\left(\bigcap_{i=1}^{t} \{x_i \to \partial B(L)\}\right) \pi_k$$

Hence, for $k \geqslant 4$ we obtain

$$P_p\left(\bigcap_{i=1}^{t+1} \{x_i \to \partial B(L)\}\right) \leqslant C_\varepsilon^{-1} \tilde{C}_\varepsilon \pi_k P_p\left(\bigcap_{i=1}^{t} \{x_i \to \partial B(L)\}\right)$$

For $k \leqslant 4$ the above inequality is obvious. Since there are at most $8t(k+1)$

points at distance k from $\{x_1,..., x_t\}$ and since there are at most $8(L-k+1)$ points at distance k from the boundary $\partial B(L)$, we have

$$M_{t+1}[L(p)] \leqslant \left[C_\varepsilon \sum_{k=0}^{2L(p)} 8t(k+1)\,\pi_k + \tilde{C}_\varepsilon \sum_{k=0}^{2L(p)} 8(L-k+1)\,\pi_k \right] \times \sum_{x_1,...,x_t \in B(L)} P_p\left(\bigcap_{i=1}^{t} \{x_i \to \partial B(L)\} \right)$$

Since the first two sums involving π_k are bounded by const $\cdot\,(t+1)\,L(p)$ $\sum_{k=0}^{2L(p)} \pi_k$ and the last sum is exactly $M_t[L(p)]$, (2.3) is obtained.

Remark. Kesten[4] further showed

$$\sum_{k=0}^{2L(p)} \pi_k \asymp \sum_{k=0}^{L(p)} \pi_k \asymp L(p)\,\pi_p(L) \tag{2.4}$$

Hence, (2.3) implies

$$M_{t+1}[L(p)] \leqslant C_\varepsilon(t+1)\,L(p)^2\,\pi_p(L)\,M_t[L(p)] \tag{2.5}$$

Note that

$$M_1[L(p)] = \sum_{x \in B(L)} P_p(x \to \partial B(L)) \leqslant C_\varepsilon \sum_{k=0}^{L(p)} (L-k+1)\,\pi_k \leqslant K_1 L^2 \pi_p(L)$$

This shows

$$M_{t+1}[L(p)] \leqslant (t+1)!\,[K_1 L^2(p)\,\pi_p(L)]^{t+1} \tag{2.6}$$

where K_1 is some positive constant depending only on ε.

Before leaving this section, we note that by the same argument we can show, for $t \geqslant 1$,

$$E_p\{|W_0 \cap B(L)|^t\,|\,0 \to \partial B(L)\} \leqslant C_\varepsilon(t+1)\,L^2\pi_p(L)\,E_p\{|W_0 \cap B(L)|^{t+1}\,|\,0 \to \partial B(L)\} \tag{2.7}$$

and

$$E_p\{|W_0 \cap B(L)|^t\,|\,0 \to \partial B(L)\} \leqslant (t+1)!\,[K_2 L^2 \pi_p(L)]^t \tag{2.8}$$

where K_2 is some positive constant depending only on ε. The inequalities (2.6) and (2.8) play important roles in the proof of (1.1). For a proof of this see Ref. 5, Section 3.

3. PROOF OF PROPOSITION 1

In this section we show Proposition 1. Throughout we take $p < p_c$. First we claim

$$S_1(p) \leqslant C_\varepsilon S_L(p) \tag{3.1}$$

The proof given here was communicated to the author by H. Kesten. To prove this, it is enough to show there exist constants C_1, C_2, $C_3 > 0$, so that

$$P_p(|W_0| \geqslant C_1 k L^2 \pi_p(L)) \leqslant C_2 \exp(-C_3 k) \tag{3.2}$$

We denote by $B(\underset{\sim}{n})$ the box of size $L(p, \varepsilon)$ centered at $\underset{\sim}{n} = (n_1 2L, n_2 2L)$, $(n_1, n_2) \in Z^2$, i.e., $B(\underset{\sim}{n}) = \{x \in Z^2 \colon |x - \underset{\sim}{n}| \leqslant L(p, \varepsilon)\}$. We say that $\underset{\sim}{n}$ is connected to $\underset{\sim}{0}$ if there is an occupied path connecting $B(\underset{\sim}{n})$ and $B(\underset{\sim}{0})$. Let $\Gamma = \{\underset{\sim}{n} \colon \underset{\sim}{0} \to \underset{\sim}{n}\}$. It can be seen from the proof of Theorem 5.1 of Ref. 3 that

$$P_p(|\Gamma| > k) \leqslant \tilde{C}_2 \exp(-\tilde{C}_3 k) \tag{3.3}$$

for some positive constants $\tilde{C}_2$, $\tilde{C}_3$ provided ε is sufficiently small. Thus, to prove (3.2), we need to show the exponential decay in k of $P_p(|W_0| \geqslant C_1 k L^2 \pi_p(L); |\Gamma| \leqslant k)$.

Let

$$Z_{\underset{\sim}{n}_i} = \# \{x \in B(\underset{\sim}{n}_i) \colon x \to \partial B(\underset{\sim}{n}_i)\}$$

We have

$$\begin{aligned}
&P_p(|W_0| \geqslant C_1 k L^2 \pi_p(L); |\Gamma| \leqslant k) \\
&\quad \leqslant \sum_{\Gamma \colon |\Gamma| \leqslant k} P_p\left(\sum_{\underset{\sim}{n}_i \in \Gamma} Z_{\underset{\sim}{n}_i} \geqslant C_1 k L^2 \pi_p(L)\right) \\
&\quad \leqslant \sum_{\Gamma \colon |\Gamma| \leqslant k} \inf_{r>0} \left(\{\exp[-r C_1 k L^2 \pi_p(L)]\} E_p \exp\left(\sum_{\underset{\sim}{n}_i \in \Gamma} r Z_{\underset{\sim}{n}_i}\right)\right) \\
&\quad \leqslant \sum_{\Gamma \colon |\Gamma| \leqslant k} \inf_{r>0} (\{\exp[-r C_1 k L^2 \pi_p(L)]\} [E_p \exp(r Z_{\underset{\sim}{n}_1})]^k)
\end{aligned}$$

Use (2.6) to get

$$\begin{aligned}
E_p \exp(r Z_{\underset{\sim}{n}_1}) &= \sum_{t=0}^{\infty} \frac{r^t}{t!} E_p(Z^t_{\underset{\sim}{n}_1}) \\
&\leqslant \sum_{t=0}^{\infty} \frac{r^t}{t!} (t+1)! \, [K_1 L^2 \pi_p(L)]^t \\
&= \sum_{t=0}^{\infty} r^t (t+1) [K_1 L^2 \pi_p(L)]^t
\end{aligned}$$

Hence, by choosing $r = 1/2K_1 L^2\pi_p(L)$, we have

$$E_p \exp(rZ_{\underline{n}1}) \leqslant \sum_{t=0}^{\infty} (t+1)(1/2)^t = C_5 < \infty$$

Since the number of clusters Γ with $|\Gamma| \leqslant k$ is bounded by C_4^k for some positive constant C_4,

$$P_p(|W_0| \geqslant C_1 kL^2\pi_p(L); |\Gamma| \leqslant k) \leqslant C_4^k[\exp(-C_1 k/2K_1)]\ C_5^k$$

We choose $C_1 = 4K_1 \log(C_4 C_5)$ to obtain (3.2).

Thus, from (3.1) the critical exponent of $S_{\mathrm{I}}(p)$ is not larger than Δ. To get the other bound, we consider

$$E_p(|W_0|^t) = \sum_{n=1}^{\infty} n^t P_p(|W_0| = n)$$

But by the supermultiplicative property of $n^{-1}P_p(|W_0| = n)$

$$P_p(|W_0| = n) \leqslant n \exp[-n/S_{\mathrm{I}}(p)]$$

Hence

$$E_p(|W_0|^t) \leqslant \sum_{n=1}^{\infty} n^t n \exp[-n/S_{\mathrm{I}}(p)] \leqslant KS_{\mathrm{I}}(p)^{t+2}$$

where $K = K(t)$ is some positive constant. But Kesten[5] showed that

$$E_p(|W_0|^t) \geqslant C_t S_L(p)^t\, \pi_p(L)$$

where C_t is some constant depending on t. Then

$$C_t S_L(p)^t\, \pi_p(L) \leqslant KS_{\mathrm{I}}(p)^{t+2}$$

Hence,

$$-\frac{\log S_L(p)}{\log|p-p_c|} - \frac{1}{t}\frac{\log(C_t \pi_p(L))}{\log|p-p_c|} \leqslant -\frac{t+2}{t}\frac{\log(KS_{\mathrm{I}}(p))}{\log|p-p_c|}$$

Letting $p \uparrow p_c$ and then $t \to \infty$, we obtain the result that $S_{\mathrm{I}}(p) \approx |p-p_c|^{-\Delta}$.

4. PROOF OF PROPOSITION 2

The proof of Proposition 2 will be based on Kesten's theorem. In fact, on one hand, using the Cauchy–Schwartz inequality and then (1.3), we have

$$\left\{\sum_{\infty > n \geqslant \delta S_L(p)} nP_p(|W_0| = n)\right\}\left\{\sum_{\infty > n \geqslant \delta S_L(p)} \frac{1}{n} P_p(|W_0| = n)\right\}$$
$$\geqslant \left\{\sum_{\infty > n \geqslant \delta S_L(p)} P_p(|W_0| = n)\right\}^2 \geqslant \left\{\frac{1}{2}\pi_p(L)\right\}^2$$

Thus,

$$\begin{aligned} f_s(p) &\equiv \sum_{n \geqslant \delta S_L(p)} n^{-1} P_p(|W_0| = n) \\ &\geqslant \tfrac{1}{4}\pi_p^2(L) \Big/ \sum_{\infty > n \geqslant \delta S_L(p)} nP_p(|W_0| = n) \\ &\geqslant \tfrac{1}{4}\pi_p^2(L)/E_p(|W_0|; |W_0| < \infty) \\ &\geqslant \tfrac{1}{4}\pi_p^2(L)/C_\varepsilon \pi_p^2(L)\, L^2(p) \\ &\geqslant 1/[4C_\varepsilon L^2(p)] \end{aligned}$$

where in the last but one inequality we used (1.2). On the other hand,

$$\begin{aligned} f_s(p) &\leqslant \frac{1}{[\delta S_L(p)]^2} \sum_{\infty > n \geqslant \delta S_L(P)} nP_p(|W_0| = n) \\ &\leqslant \frac{1}{\delta^2 S_L^2(p)} E_p(|W_0|; |W_0| < \infty) \\ &\leqslant \frac{C_\varepsilon L^2(p)\, \pi_p^2(L)}{\tilde{C}_\varepsilon \delta^2 [L^2(p)\, \pi_p(L)]^2} = \frac{C_\varepsilon}{\tilde{C}_\varepsilon \delta^2 L^2(p)} \end{aligned}$$

by (1.1) and (1.2). This shows Proposition 2.

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